High-Order Eigenfunctions of the Graetz Problem

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The Graetz problem is an analysis of heat transfer between a fluid moving at steady state, laminar conditions and a tube wall maintained at a fixed temperature. Solutions to this problem give the fluid temperature as functions of the distance along the conduit and the position within the conduit. At the beginning of the conduit, $x = 0$, the fluid has a uniform temperature and a fully developed laminar velocity profile. The analogous case for a fluid flowing between parallel plates is also treated. The abundant literature on this problem is referenced by Brown (*1*) in a recent paper and will not be listed here. The purpose of this communication is to extend the tabulation of eigenfunctions contained in the solutions.

The solution for the case of circular ducts is

$$\frac{t_w - t}{t_w - t_o} = \sum_{n=1}^{\infty} C_n Y_n \exp\left(\frac{-\lambda_n^2 x}{R\, N_{Pe}}\right) \qquad (1)$$

and for flat ducts

$$\frac{t_w - t}{t_w - t_o} = \sum_{f=1}^{\infty} C_f Y_f \exp\left(\frac{-8\lambda_f^2 x}{3\, R\, N_{Pe}}\right) \qquad (2)$$

Values for the coefficients C and the eigenvalues λ to order ten for flat ducts and eleven for circular ducts are tabulated (*1*). Higher-order values may be estimated by means of asymptotic formulas (*4*). The first six eigenfunctions are also published (*1*).

The eigenfunctions Y_n and Y_f may be related to the hypergeometric function (*2, 3*):

$$Y_n\,(r/R) = e^{\frac{-\lambda_n}{2}(r/R)^2}\ {}_1F_1\left(\frac{1}{2} - \frac{\lambda_n}{4};\ 1;\ \frac{\lambda_n r^2}{R^2}\right) \qquad (3)$$

$$Y_f\,(r/R) = e^{\frac{-\lambda_f}{2}(r/R)^2}\ {}_1F_1\left(\frac{1}{4} - \frac{\lambda_f}{4};\ \frac{1}{2};\ \frac{\lambda_f r^2}{R^2}\right) \qquad (4)$$

The hypergeometric functions can be evaluated conveniently on a digital computer by means of power-series expansions. Such expansions appear to converge more rapidly and provide more accurate results, with a fixed number of significant figures, than those employed by Brown.

Tables 1 and 2 give the eigenfunctions at increments in r/R of 0.1 for functions to order 15. Since $Y\,(0)$ is 1 and $Y\,(1)$ is 0, these were not tabulated. Because these functions oscillate rapidly interpolation is impossible. The computations were performed on a Datatron computer with double precision floating point arithmetic. Seventeen significant figures were carried.

NOTATION

${}_1F_1$ = confluent hypergeometric function

N_{Pe} = Peclet number for tubes $N_{Pe} = \frac{2R\,U_m}{\alpha}$; for flat plates $N_{Pe} = \frac{4U_m R}{\alpha}$

r = distance from center of duct

R = tube radius or half width of flat duct

t = temperature of the fluid at any point

t_o = temperature of the fluid at $x = 0$

t_w = temperature of the conduit wall, a constant

U_m = mean fluid velocity

x = distance down the conduit measured from the point at which the fluid has a uniform temperature t_o

α = thermal diffusivity of the fluid

LITERATURE CITED

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2. Lauwerier, H. A., *App. Sci. Res.*, **A2**, 184 (1951).
3. Schenk, J., and J. M. Dumoré, *ibid.*, **A4**, 39 (1953).
4. Sellars, J. R., Myron Tribus, and J. S. Klein, *Trans. Am. Soc. Mech. Engrs.*, **78**, 441 (1956).

TABLE 1. VALUES FOR THE EIGENFUNCTION $Y_n\ (r/R)$
(Circular Ducts)

n \ r/R	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
6	+0.07488	−0.32122	+0.28982	−0.04766	−0.20532	+0.19750	+0.10372	−0.20893	−0.19522
7	−0.12642	−0.07613	+0.20122	−0.25168	+0.19395	−0.01391	−0.18883	+0.14716	+0.19927
8	−0.28107	+0.17716	−0.10751	+0.03452	+0.05514	−0.15368	+0.20290	−0.07985	−0.20068
9	−0.37523	+0.29974	−0.25305	+0.22174	−0.20502	+0.19303	−0.15099	+0.01298	+0.19967
10	−0.40326	+0.23915	−0.08558	−0.02483	+0.08126	−0.09176	+0.05652	+0.04787	−0.19645
11	−0.36817	+0.04829	+0.16645	−0.20058	+0.13289	−0.06474	+0.04681	−0.09797	+0.19120
12	−0.28088	−0.15310	+0.19847	+0.01714	−0.15931	+0.16099	−0.12577	+0.13375	−0.18409
13	−0.15836	−0.24999	−0.00845	+0.18456	−0.01927	−0.13393	+0.15742	−0.15311	+0.17527
14	−0.02118	−0.19545	−0.18955	−0.01074	+0.15967	+0.01258	−0.13539	+0.15549	−0.16491
15	+0.10953	−0.03182	−0.13083	−0.17183	−0.08560	+0.10927	+0.07069	−0.14189	+0.15319

TABLE 2. VALUES FOR THE EIGENFUNCTION $Y_f\ (r/R)$
(Flat Ducts)

f \ r/R	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
6	−0.5606	−0.3986	+1.0161	−0.5737	−0.6272	+1.0360	+0.3505	−1.1651	−1.0720
7	−0.8396	+0.3828	+0.2708	−0.8814	+1.0315	−0.3214	−0.9217	+0.9305	+1.1956
8	−0.9865	+0.9352	−0.8106	+0.5467	−0.0632	−0.6347	+1.1775	−0.5942	−1.2988
9	−0.9783	+0.9279	−0.8863	+0.8980	−0.9893	+1.1146	−1.0311	+0.1934	+1.3808
10	−0.8162	+0.3651	+0.1377	−0.5198	+0.7263	−0.7588	+0.5331	+0.2283	−1.4411
11	−0.5257	−0.4163	+0.9909	−0.9136	+0.5023	−0.1659	+0.1465	−0.6254	+1.4792
12	−0.1526	−0.9485	+0.6144	+0.4927	−1.0632	+0.9663	−0.7763	+0.9556	−1.4951
13	+0.2446	−0.9127	−0.5246	+0.9282	+0.2110	−1.0428	+1.1420	−1.1835	+1.4888
14	+0.6034	−0.3304	−1.0125	−0.4653	+0.9217	+0.3382	−1.1194	+1.2849	−1.4607
15	+0.8672	+0.4497	−0.2437	−0.9419	−0.8295	+0.6197	+0.7162	−1.2492	+1.4113